

On connected automorphism groups of algebraic varieties

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Abstract

Let X be a normal projective algebraic variety, G its largest connected automorphism group, and $A(G)$ the Albanese variety of G . We determine the isogeny class of $A(G)$ in terms of the geometry of X . In characteristic 0, we show that the dimension of $A(G)$ is the rank of every maximal trivial direct summand of the tangent sheaf of X . Also, we obtain an optimal bound for the dimension of the largest anti-affine closed subgroup of G (which is the smallest closed subgroup that maps onto $A(G)$).

1 Introduction

Throughout this note, we consider algebraic varieties over a fixed algebraically closed field k . Let X be a complete variety; then its automorphism functor is represented by a group scheme $\mathrm{Aut}(X)$, locally of finite type (see [MO67, Thm. 3.7]). Thus, X has a largest connected algebraic group $G = G(X)$ of automorphisms: the reduced neutral component of $\mathrm{Aut}(X)$. In general, G is not affine; equivalently, it is not linear. For instance, if X is an abelian variety, then G is just X acting on itself by translations.

How to measure the nonlinearity of G in terms of the geometry of X ? In this note, we obtain several partial answers to that question. We begin by determining the Albanese variety of G up to isogeny. For this, we recall a theorem of Chevalley (see [Ch60], and [Co02], [BSU12, Chap. 2] for modern proofs): G sits in an exact sequence of connected algebraic groups

$$1 \longrightarrow G_{\mathrm{aff}} \longrightarrow G \xrightarrow{\alpha} A(G) \longrightarrow 1,$$

where G_{aff} is affine and $A(G)$ is an abelian variety; the map α is the Albanese morphism of G . We also need the following:

Definition 1.1. A *homogeneous fibration* is a morphism $f : X \rightarrow A$ satisfying the following conditions:

- (i) $f_*(\mathcal{O}_X) = \mathcal{O}_A$.
- (ii) A is an abelian variety.
- (iii) f is isomorphic to its pull-back by any translation in A .

Note that every fiber of a homogeneous fibration is connected by (i), and all these fibers are isomorphic by (iii). We may now state our first result:

Proposition 1.2. *Let X be a normal projective variety.*

(i) *For any homogeneous fibration $f : X \rightarrow A$, the G -action on X induces a transitive action of $A(G)$ on A by translations.*

(ii) *There exists a homogeneous fibration $f : X \rightarrow A$ such that the resulting homomorphism $A(G) \rightarrow A$ is an isogeny.*

The (easy) proof is given in Subsection 2.1. The statement also holds when X is nonsingular, but fails for certain normal threefolds in view of Example 2.3.3 (based on a construction of Raynaud, see [Ra70, XIII 3.2]). Also, there generally exists no homogeneous fibration $f : X \rightarrow A(G)$, as shown by Example 2.3.4.

Next, we obtain an interpretation of the Lie algebra $\mathrm{Lie}(G_{\mathrm{aff}})$ in terms of nowhere vanishing vector fields, under the assumption that k has characteristic 0 (but X is possibly singular). Then $\mathrm{Lie}(G)$ is identified with the Lie algebra of vector fields, that is, of global sections of the sheaf \mathcal{T}_X of derivations of \mathcal{O}_X (see again [MO67, Thm. 3.7]). The associated evaluation map

$$\mathrm{op}_X : \mathcal{O}_X \otimes_k \mathrm{Lie}(G) \longrightarrow \mathcal{T}_X$$

yields a map

$$\mathrm{op}_{X,x} : \mathrm{Lie}(G) \longrightarrow T_x(X), \quad \xi \longmapsto \xi_x$$

for any point $x \in X$, where $T_x(X) := \mathcal{T}_X \otimes_{\mathcal{O}_X} k(x)$ denotes the Zariski tangent space of X at x . We say that x is a zero of ξ , if $\xi_x = 0$. Since $\mathrm{op}_{X,x}$ is identified to the differential of the orbit map $G \rightarrow X$, $g \mapsto g \cdot x$ at the neutral element of G , we see that $\xi_x = 0$ if and only if ξ sits in the isotropy Lie algebra $\mathrm{Lie}(G)_x$.

Proposition 1.3. *Assume that $\mathrm{char}(k) = 0$. Let X be a complete normal variety and let V be a subspace of $\mathrm{Lie}(G)$. Then the following assertions are equivalent:*

- (i) *$V \setminus \{0\}$ consists of nowhere vanishing vector fields.*
- (ii) *$V \cap \mathrm{Lie}(G_{\mathrm{aff}}) = \{0\}$.*
- (iii) *op_X identifies $\mathcal{O}_X \otimes V$ to a direct summand of \mathcal{T}_X .*

This result is proved in Subsection 2.2. It implies readily that the maximal direct summands of \mathcal{T}_X which are trivial (i.e., direct sums of copies of \mathcal{O}_X) are exactly the $\mathcal{O}_X \otimes_k V$, where V is a subspace of $\mathrm{Lie}(G)$ such that $V \oplus \mathrm{Lie}(G_{\mathrm{aff}}) = \mathrm{Lie}(G)$. We may find such subspaces which are abelian Lie subalgebras since $G = Z(G)G_{\mathrm{aff}}$, where $Z(G)$ denotes the center of G (see e.g. [BSU12, Prop. 3.1.1]). But in general, there exists no closed subgroup H of G such that $\mathrm{Lie}(H) \oplus \mathrm{Lie}(G_{\mathrm{aff}}) = \mathrm{Lie}(G)$, as shown by Examples 2.3.1 and 2.3.2. A full decomposition of the tangent sheaf is obtained in [GKP11] for projective varieties with canonical singularities and numerically trivial canonical class.

As another consequence of Proposition 1.3, a vector field $\xi \in \mathrm{Lie}(G)$ has a zero if and only if $\xi \in \mathrm{Lie}(G_{\mathrm{aff}})$. In particular, ξ has no zero if it sits in the Lie algebra of some abelian subvariety of G . But the converse to the latter statement does not hold: actually, 2.3.1 and 2.3.2 provide examples of nowhere vanishing vector fields ξ on a projective nonsingular surface X , such that ξ does not sit in the Lie algebra of any abelian variety acting on X , nor on any finite étale cover of X . This shows that some assumption is missing in a statement attributed to D. Lieberman in [Pe11], and in Conjecture 4.24 there; for example, X is not uniruled (then G is an abelian variety in view of Chevalley's theorem).

Finally, we consider another way to measure the nonlinearity of G , by bounding the dimension of its largest anti-affine subgroup in terms of the dimension of X . Recall that an algebraic group H is called anti-affine, if any global regular function on H is constant. By a result of Rosenlicht (see [Ro56], and [BSU12, Sec. 3.2] for a modern proof), G has a largest anti-affine subgroup G_{ant} which is also the smallest closed subgroup mapped onto $A(G)$ by α ; moreover, G_{ant} is connected and contained in the center $Z(G)$. In particular, G is linear if and only if G_{ant} is trivial.

Actually, we bound the dimension of a connected algebraic group of automorphisms of a possibly non-complete variety X , in arbitrary characteristic. Then it is known that X has a largest anti-affine group of automorphisms (see e.g. [BSU12, Prop. 5.5.4]) and the proof given there shows that the dimension of this group is at most $3 \dim(X)$. Yet this bound is far from being sharp:

Theorem 1.4. *Let X be a variety of dimension n , and G an anti-affine algebraic group of automorphisms of X . Then*

$$\dim(G) \leq \begin{cases} \max(n, 2n - 4) & \text{if } \text{char}(k) = 0, \\ n & \text{if } \text{char}(k) > 0, \end{cases}$$

and this bound is optimal in both cases.

The proof is given in Section 3; it yields some information on the varieties for which this bound is attained. In positive characteristic, these extremal varieties are just equivariant compactifications of semiabelian varieties, also called semiabelic varieties (see [Al02]). The case of characteristic 0 turns out to be more involved, and we do not obtain a full description of all extremal varieties; examples are presented in the final subsection.

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2 Proofs of Propositions 1.2 and 1.3

2.1 Proof of Proposition 1.2

(i) follows readily from a variant of a result of Blanchard (see [Bl56, Prop. I.1] and [BSU12, Prop. 4.2.1]): let $\varphi : Y \rightarrow Z$ be a proper morphism of varieties such that $\varphi_*(\mathcal{O}_Y) = \mathcal{O}_Z$ and let H be a connected algebraic group acting on Y . Then there is a unique action of H on Z such that φ is equivariant. Indeed, by that result, the G -action on X induces an action on A such that f is equivariant. This yields a homomorphism of algebraic groups $\beta : G \rightarrow A$ such that G acts on A by translations via β , and in turn a homomorphism $\gamma : A(G) \rightarrow A$. To complete the proof, it suffices to show that γ , or equivalently β , is surjective. But since f is isomorphic to its pull-backs under all translations of A , we see that these translations lift to automorphisms of X . In other words, denoting by $\text{Aut}(X, f)$ the group scheme of automorphisms of X preserving f and by $f_* : \text{Aut}(X, f) \rightarrow \text{Aut}(A)$ the natural homomorphism, the image of f_* contains the subgroup A of translations.

Since G is the reduced connected component of $\text{Aut}(X, f)$ and $f_*|_G = \beta$, it follows that $\beta(G) = A$.

(ii) By [Br10, Thm. 2] (which generalizes a theorem of Nishi and Matsumura, see [Ma63]), there exists a G -equivariant morphism $\varphi : X \rightarrow B$, where B is an abelian variety, quotient of $A(G)$ by a finite subgroup scheme. Consider the Stein factorization $\varphi = \psi \circ f$, where $f : X \rightarrow A$ satisfies $f_*(\mathcal{O}_X) = \mathcal{O}_A$, and $\psi : A \rightarrow B$ is finite. Then both f and ψ are G -equivariant. Thus, the variety A is a unique G -orbit. It follows that A is also an abelian variety, quotient of $A(G)$ by a finite subgroup scheme. So f is the desired homogeneous fibration.

If X is assumed to be nonsingular (instead of projective and normal), then the assertion follows from the above-mentioned theorem of Nishi and Matsumura: let H be a connected algebraic group of automorphisms of a nonsingular variety Y . Then there exists an H -equivariant morphism $\varphi : Y \rightarrow A$, where A is the quotient of $A(H)$ by some finite subgroup scheme. For a modern proof of that theorem, see [Br10].

2.2 Proof of Proposition 1.3

(i) \Rightarrow (ii) Let $\xi \in \text{Lie}(G_{\text{aff}})$. Then ξ sits in the Lie algebra of some Borel subgroup B of G_{aff} (see [Bo91, 14.16]). By Borel's fixed point theorem, B fixes some point $x \in X$, which yields the desired zero of ξ .

(ii) \Rightarrow (iii) We may replace V with any larger subspace of $\text{Lie}(G)$ that satisfies (ii), and hence assume that $V \oplus \text{Lie}(G_{\text{aff}}) = \text{Lie}(G)$. Equivalently, the differential of $\alpha : G \rightarrow A(G)$ restricts to an isomorphism $V \xrightarrow{\cong} \text{Lie}(A(G))$.

Consider the nonsingular locus U of X . This is an open G -stable subset of X ; hence there exists a G -equivariant morphism $\varphi : U \rightarrow A(G)/F$ for some finite subgroup F of $A(G)$. By equivariance, the morphism φ is smooth, and hence yields a surjective \mathcal{O}_U -linear map

$$d\varphi : \mathcal{T}_U \longrightarrow \varphi^*(\mathcal{T}_{A(G)/F}).$$

But $\varphi^*(\mathcal{T}_{A(G)/F}) \cong \mathcal{O}_U \otimes_k \text{Lie}(A(G)/F) \cong \mathcal{O}_U \otimes_k \text{Lie}(A(G))$ and hence we may view $d\varphi$ as an \mathcal{O}_U -linear map $\mathcal{T}_U \rightarrow \mathcal{O}_U \otimes_k \text{Lie}(A(G))$. Also, denoting by $\psi : \mathcal{O}_U \otimes_k V \rightarrow \mathcal{T}_U$ the restriction of op_U , the map $d\varphi \circ \psi : \mathcal{O}_U \otimes_k V \rightarrow \mathcal{O}_U \otimes_k \text{Lie}(A(G))$ is an isomorphism in view of our assumption on V . Thus, ψ identifies $\mathcal{O}_U \otimes_k V$ to a direct summand of \mathcal{T}_U .

To obtain the analogous assertion on X , just note that the sheaf $\mathcal{T}_X = \text{Hom}_X(\Omega_X^1, \mathcal{O}_X)$ is reflexive; hence $\mathcal{T}_X = i_*(\mathcal{T}_U)$, where $i : U \rightarrow X$ denotes the inclusion. Moreover, every direct sum decomposition of \mathcal{T}_U yields a direct sum decomposition of \mathcal{T}_X .

(iii) \Rightarrow (i) The assumption implies that the map $V \rightarrow T_x(X)$, $\xi \mapsto \xi_x$ is injective for any $x \in X$. Hence every $\xi \in V \setminus \{0\}$ has no zero.

2.3 Examples

In this subsection, E denotes an elliptic curve. We first present (after [BSU12, Ex. 4.2.4]) two examples of ruled surfaces X over E for which the connected automorphism group G is neither linear, nor an abelian variety. The description of G in both cases follows from the classification of the automorphism group schemes of all ruled surfaces by Maruyama (see [Ma71, Thm. 3]).

2.3.1 Let V be a vector bundle of rank 2 on E , obtained as a nonsplit extension of the trivial line bundle \mathcal{O}_E by \mathcal{O}_E . Let

$$\pi : X := \mathbb{P}(V) \longrightarrow E$$

be the associated ruled surface. Then G sits in an exact sequence of algebraic groups

$$1 \longrightarrow \mathbb{G}_a \longrightarrow G \xrightarrow{\alpha} E \longrightarrow 1,$$

where \mathbb{G}_a denotes the additive group. Moreover, G acts on X with two orbits: the obvious section of π , which is a closed orbit isomorphic to G/\mathbb{G}_a , and its complement, an open orbit isomorphic to G . The class of the above extension in $\text{Ext}^1(E, \mathbb{G}_a) \cong H^1(E, \mathcal{O}_E) \cong \text{Ext}^1(\mathcal{O}_E, \mathcal{O}_E)$ is identified to the class of the extension $0 \rightarrow \mathcal{O}_E \rightarrow V \rightarrow \mathcal{O}_E \rightarrow 0$.

We claim that there exists no closed subgroup H of G such that $\text{Lie}(H) \oplus \text{Lie}(\mathbb{G}_a) = \text{Lie}(G)$. Otherwise, the scheme-theoretic intersection $H \cap \mathbb{G}_a$ is a finite reduced subgroup scheme of \mathbb{G}_a , and hence is trivial. It follows that H is isomorphic to E via α ; thus, the multiplication of G restricts to an isomorphism $H \times \mathbb{G}_a \cong G$. But this contradicts the fact that G is a nonsplit extension of E by \mathbb{G}_a .

Next, assume that $\text{char}(k) = 0$ and let $\xi \in \text{Lie}(G)$ be a nowhere vanishing vector field, i.e., $\xi \notin \text{Lie}(\mathbb{G}_a)$. We claim that there is no finite étale morphism $f : \tilde{X} \rightarrow X$ such that ξ (viewed as a vector field on \tilde{X}) sits in the Lie algebra of some abelian variety A acting on \tilde{X} . Otherwise, by the Nishi-Matsumura theorem again, we obtain an A -equivariant morphism $\varphi : \tilde{X} \rightarrow A/F$, where F is a finite subgroup of A ; using the Stein factorization, we may further assume that the fibers of φ are connected. Let Y be the fiber at the origin; then Y is stable by F and the map $A \times Y \rightarrow \tilde{X}$, $(a, y) \mapsto a \cdot y$ factors through an A -equivariant isomorphism $A \times^F Y \xrightarrow{\cong} \tilde{X}$, where $A \times^F Y$ denotes the quotient of $A \times Y$ by the action of F via $g \cdot (a, y) := (a - g, g \cdot y)$. We may thus replace \tilde{X} with its finite étale cover $A \times Y$. Now f restricts to a finite étale morphism $f^{-1}(\pi^{-1}(z)) \rightarrow \pi^{-1}(z)$ for any $z \in E$. Since $\pi^{-1}(z) \cong \mathbb{P}^1$, it follows that \tilde{X} is covered by copies of \mathbb{P}^1 , and hence $Y \cong \mathbb{P}^1$. Thus, A is an elliptic curve, and the projection $p : \tilde{X} = A \times \mathbb{P}^1 \rightarrow A$ is the Albanese morphism. Since $\pi : X \rightarrow E$ is the Albanese morphism as well, there exists a unique morphism $\varphi : A \rightarrow E$ such that the square

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{p} & A \\ f \downarrow & & \varphi \downarrow \\ X & \xrightarrow{\pi} & E \end{array}$$

is commutative. Clearly, φ is finite and surjective, hence an isogeny of elliptic curves. Thus,

$$\varphi^* : \text{Ext}^1(\mathcal{O}_E, \mathcal{O}_E) = H^1(E, \mathcal{O}_E) \longrightarrow H^1(A, \mathcal{O}_A) = \text{Ext}^1(\mathcal{O}_A, \mathcal{O}_A)$$

is an isomorphism. In particular, the vector bundle $\varphi^*(V)$ is a nonsplit extension of \mathcal{O}_A by itself. Moreover, the above commutative square yields a morphism

$$\psi : A \times \mathbb{P}^1 = \tilde{X} \longrightarrow X \times_E A = \mathbb{P}(\varphi^*(V))$$

which lifts the identity of A . Since f factors through ψ , we see that ψ is finite and étale. It follows that ψ is an isomorphism, and hence that $\varphi^*(V) \cong L \oplus L$ for some line bundle L on A . But $\varphi^*(V)$ is indecomposable, a contradiction.

2.3.2 Consider the rank 2 vector bundle $V := L \oplus \mathcal{O}_E$, where L is a line bundle of degree 0 on E . Then for the associated ruled surface $\pi : X \rightarrow E$, we have an exact sequence of algebraic groups

$$1 \longrightarrow \mathbb{G}_m \longrightarrow G \xrightarrow{\alpha} E \longrightarrow 1,$$

where \mathbb{G}_m denotes the multiplicative group. Moreover, G acts on X with three orbits: the two obvious sections of π , which are closed orbits isomorphic to G/\mathbb{G}_m , and their complement, an open orbit isomorphic to G . The class of the above extension in $\text{Ext}^1(E, \mathbb{G}_m) \cong \text{Pic}^o(E) \cong E$ is identified to the class of L .

Assume that L has infinite order, i.e., the extension giving G is not split by any isogeny $A \rightarrow E$. Then, arguing as in the above example, one checks that there exists no closed subgroup H of G such that $\text{Lie}(H) \oplus \text{Lie}(\mathbb{G}_m) = \text{Lie}(G)$. Moreover, in characteristic 0, given $\xi \in \text{Lie}(G) \setminus \text{Lie}(\mathbb{G}_m)$, there is no finite étale morphism $f : \tilde{X} \rightarrow X$ such that ξ sits in the Lie algebra of some abelian variety acting on \tilde{X} .

2.3.3 Next, we construct a complete normal threefold X equipped with a faithful action of the elliptic curve E and having a trivial Albanese variety.

By [Br10, Ex. 6.4] (after [Ra70, XIII 3.2]), there exists a normal affine surface Y having exactly two singular points y_1, y_2 and equipped with a morphism

$$f : \dot{E} \times \dot{\mathbb{A}}^1 \longrightarrow Y,$$

where $\dot{E} := E \setminus \{0\}$ and $\dot{\mathbb{A}}^1 := \mathbb{A}^1 \setminus \{0\}$, such that $f(\dot{E} \times \{1\}) = \{y_1\}$, $f(\dot{E} \times \{-1\}) = \{y_2\}$ and the restriction of f to the complement, $\dot{E} \times (\mathbb{A}^1 \setminus \{1, -1\})$, is an open immersion.

There exists a normal projective surface Z containing Y as a dense open subset and having the same singular locus, $\{y_1, y_2\}$. Let $V_1 := Z \setminus \{y_2\}$, $V_2 := Z \setminus \{y_1\}$ and $V_{12} := V_1 \cap V_2$. Then V_{12} is nonsingular and contains $\dot{E} \times (\mathbb{A}^1 \setminus \{1, -1\})$ as an open subset. So the projection of that subset to \dot{E} extends to a dominant morphism

$$p : V_{12} \longrightarrow E.$$

We may glue $E \times V_1$ and $E \times V_2$ along $E \times V_{12}$ via the automorphism of $E \times V_{12}$ given by

$$(x, y) \longmapsto (x + p(y), y).$$

This yields a variety X equipped with an action of E (by translations on the first factor of each $E \times V_i$) and with an E -equivariant morphism

$$\pi : X \longrightarrow Z$$

which extends the projections $E \times V_i \rightarrow V_i$. Clearly, π is a principal E -bundle, locally trivial for the Zariski topology. In particular, π is proper, and hence X is complete; also, X is normal since so is Z .

We claim that the Albanese variety of V_1 is trivial. Consider indeed a morphism $\alpha : V_1 \rightarrow A$, where A is an abelian variety. Then the restriction of α to $\dot{E} \times (\mathbb{A}^1 \setminus \{1, -1\})$

factors through the projection to \dot{E} . But f sends $\dot{E} \times (\mathbb{A}^1 \setminus \{-1\})$ to V_1 , and composing with α yields a constant morphism since $f(\dot{E} \times \{1\}) = \{y_1\}$. Thus, α is constant as claimed.

By that claim, the Albanese variety of Z is trivial as well. In view of Proposition 1.2, it follows that the connected automorphism group $G(Z)$ is linear. Moreover, the principal E -bundle π yields a homomorphism $\pi_* : G = G(X) \rightarrow G(Z)$ with kernel contained in the group of bundle automorphisms. The latter group is isomorphic to $\text{Hom}(Z, E)$ (see e.g. [BSU12, Rem. 6.1.5]), and hence to E . As a consequence, $E = G_{\text{ant}}$; in view of the Rosenlicht decomposition (see [BSU12, Thm. 3.2.3]), it follows that $G = EG_{\text{aff}}$, where $E \cap G_{\text{aff}}$ is finite. So the natural map $E \rightarrow A(G)$ is an isogeny.

Finally, we show that $A(X)$ is trivial. Consider again a morphism $\beta : X \rightarrow A$ to an abelian variety. Then by the claim, the restriction of β to $E \times V_1$ is of the form $(x, y) \mapsto f_1(x)$ for some morphism $f_1 : E \rightarrow A$. Likewise, we obtain a morphism $f_2 : E \rightarrow A$ such that $\beta(x, y) = f_2(x)$ for all $(x, y) \in E \times V_2$. By the construction of X , we then have $f_1(x) = f_2(x + p(y))$ for all $x \in E$ and $y \in V_2$. Thus, f_1 is constant, and so is β .

2.3.4 We now construct a complete nonsingular variety X with connected automorphism group E , which admits no homogeneous fibration to $A(G) = E$.

Choose a positive integer n , not divisible by $\text{char}(k)$. Let n_E be the multiplication by n in E , and $E[n]$ its kernel. Then $E[n] \cong (\mathbb{Z}/n\mathbb{Z})^2$ and there exists a faithful irreducible projective representation

$$\rho : E[n] \longrightarrow \text{PGL}_n.$$

Consider the associated projective bundle,

$$f : X := E \times^{E[n]} \mathbb{P}^{n-1} \longrightarrow E/E[n],$$

where $E[n]$ acts on \mathbb{P}^{n-1} via ρ . Then f is a homogeneous fibration over E identified to $E/E[n]$ via n_E . Clearly, f is the Albanese morphism of X . Also, E acts faithfully on X (since $E[n]$ acts faithfully on \mathbb{P}^{n-1}).

We claim that the resulting homomorphism $E \rightarrow G = G(X)$ is an isomorphism. To show this, it suffices to check that the kernel of the natural homomorphism $f_* : G \rightarrow E/E[n]$ is finite, or that its Lie algebra is trivial. But this Lie algebra is contained in the space of global sections of the relative tangent sheaf \mathcal{T}_f or equivalently, of its direct image $f_*(\mathcal{T}_f)$. Moreover, $f_*(\mathcal{T}_f)$ is the E -linearized sheaf on $E/E[n]$ associated with the representation of $E[n]$ on $H^0(\mathbb{P}^{n-1}, \mathcal{T}_{\mathbb{P}^{n-1}})$. Thus,

$$H^0(E/E[n], f_*(\mathcal{T}_f)) \cong (\mathcal{O}(E) \otimes H^0(\mathbb{P}^{n-1}, \mathcal{T}_{\mathbb{P}^{n-1}}))^{E[n]} \cong H^0(\mathbb{P}^{n-1}, \mathcal{T}_{\mathbb{P}^{n-1}})^{E[n]},$$

since $\mathcal{O}(E) = k$. But $H^0(\mathbb{P}^{n-1}, \mathcal{T}_{\mathbb{P}^{n-1}})$ is isomorphic to the quotient of the space of $n \times n$ matrices by the scalar matrices; this isomorphism is $E[n]$ -equivariant, where $E[n]$ acts on matrices by conjugation via ρ . Since this projective representation is irreducible, we obtain $H^0(\mathbb{P}^{n-1}, \mathcal{T}_{\mathbb{P}^{n-1}})^{E[n]} = 0$ which yields our claim.

By that claim, the action of $A(G)$ on $A(X)$ is just the action of E on $E/E[n]$ by translations. Hence X admits no homogeneous fibration to $A(G)$.

3 Proof of Theorem 1.4

3.1 In positive characteristics

Note first that we may assume X to be normal, since every connected automorphism group of X acts on its normalization. By [Br10, Thm. 1], X is then covered by G -stable quasi-projective open subsets; thus, we may further assume that X is quasi-projective.

We now consider the case where $\text{char}(k) > 0$, which turns out to be the easiest. Indeed, every anti-affine algebraic group G is a semi-abelian variety (see [BSU12, Prop. 5.4.1]), i.e., G sits in an exact sequence of connected commutative algebraic groups

$$1 \longrightarrow T \longrightarrow G \longrightarrow A \longrightarrow 1,$$

where $T = G_{\text{aff}}$ is a torus, and $A = A(G)$ an abelian variety. This yields readily:

Proposition 3.1. *Assume that $\text{char}(k) > 0$ and let G be an anti-affine group of automorphisms of a normal variety X of dimension n . Then $\dim(G) \leq n$ with equality if and only if $X \cong G \times^T Y$ for some toric T -variety Y .*

Proof. Let $x \in X$; then the isotropy subgroup scheme G_x is affine (see e.g. [BSU12, Cor. 2.1.9]). Hence the reduced neutral component of G_x is contained in T . But T_x is trivial for all x in a dense open subset U of X (see [BSU12, Lem. 5.5.5]) and hence G_x is finite for all $x \in U$. So

$$n = \dim(X) \geq \dim(G \cdot x) = \dim(G) - \dim(G_x) = \dim(G),$$

where $G \cdot x$ denotes the orbit of x . Moreover, equality holds if and only if $G \cdot x$ is open in X . Then G_x acts trivially on X since G is commutative; hence G_x is trivial.

This shows that $\dim(G) \leq n$ with equality if and only if X contains an open G -orbit with trivial isotropy subgroup scheme. Since X is normal, the latter statement is equivalent to the existence of a toric T -variety Y such that $X \cong G \times^T Y$, as follows from [Br10, Thm. 3]. \square

3.2 In characteristic 0

We begin by recalling the structure of anti-affine groups when $\text{char}(k) = 0$ (see e.g. [BSU12, Chap. 5]). Let G be a connected commutative algebraic group. Then there is an exact sequence of algebraic groups

$$1 \longrightarrow T \times U \longrightarrow G \longrightarrow A \longrightarrow 1,$$

where T is a torus, U a vector group (i.e., the additive group of a finite-dimensional vector space), and $A = A(G)$ an abelian variety. Moreover, G is anti-affine if and only if so are its quotients G/U and G/T . Note that G/U is a semi-abelian variety, and G/T is an extension of A by a vector group. Also, there is a universal such extension $E(A)$, and the corresponding vector group has the same dimension as A . Moreover, G/T is anti-affine if and only if it is a quotient of $E(A)$.

Proposition 3.2. *Assume that $\text{char}(k) = 0$ and let G be an anti-affine group of automorphisms of a normal, quasi-projective variety X of dimension n . Then*

$$\dim(G) \leq \begin{cases} n & \text{if } n \geq 4, \\ 2n - 4 & \text{if } n \leq 4, \end{cases}$$

with equality if and only if one of the following cases occurs:

(i) $X = G \times^{T \times U} Y$ for some variety Y on which $T \times U$ acts with an open orbit and a trivial isotropy group, if $n \leq 4$,

(ii) G is the universal vector extension of an abelian variety by a vector group U , and $X \cong G \times^{U \times F} S$ for some finite subgroup F of G and some (birationally) ruled surface S on which $U \times F$ acts by automorphisms preserving the ruling, if $n \geq 4$.

Proof. As in the proof of Proposition 1.2 (ii), we obtain a homogeneous fibration $\varphi : X \rightarrow B$, where B is an abelian variety, quotient of A by a finite subgroup F . Write $B = G/H$ for some closed subgroup H of G ; then H contains $G_{\text{aff}} = T \times U$ as a subgroup of finite index. Moreover, $X \cong G \times^H Y$, where Y denotes the fiber of φ at the origin of B , so that Y is a variety equipped with a faithful action of H . Since H is a commutative affine algebraic group with neutral component $T \times U$, we have $H = T \times U \times F$ for some finite subgroup F of G ; then $F \cong H/G_{\text{aff}}$.

By Lemma 3.3, Y contains a dense $T \times U$ -stable open subset Y_0 of the form $T \times Z$ for some U -stable variety Z , where $T \times U$ acts on $T \times Z$ via $(t, u) \cdot (y, z) := (ty, uz)$. We now consider two opposite special cases:

- 1) If U has an open orbit in Z , then G has an open orbit in X and hence $\dim(G) \leq n$.
- 2) If U acts trivially on Z , then U is trivial and G is a semi-abelian variety. Thus, $\dim(G) = \dim(T) + \dim(A) \leq \dim(Y) + \dim(A) = \dim(X)$, i.e., we also have $\dim(G) \leq n$.

Returning to the general case, if $\dim(Z) \leq 1$ then we are in case 1) or 2). So we may assume that

$$\dim(Z) \geq 2.$$

Since $\dim(G) = \dim(T) + \dim(U) + \dim(A)$ and $n = \dim(T) + \dim(Z) + \dim(A)$, we have $\dim(G) = n + \dim(U) - \dim(Z)$. But

$$\dim(U) \leq \dim(A)$$

and hence

$$\dim(G) \leq n + \dim(A) - \dim(Z) = 2n - \dim(T) - 2\dim(Z) \leq 2n - 4.$$

Moreover, $\dim(G) = 2n - 4$ if and only if all the displayed inequalities are equalities, i.e., $\dim(Z) = 2$, $\dim(T) = 0$ and $\dim(U) = \dim(A)$; equivalently, $Y = Z$ is a surface and G is the universal extension of A . We may further assume that the general orbits of U in Y have dimension 1: otherwise, we are again in case 1) or 2). Then there exists a dense open U -stable subset Y_1 of Y having a geometric quotient $\pi : Y_1 \rightarrow C$ by U , where C is a nonsingular curve. Replacing Y_1 with the intersection of its translates by the elements of F , we may assume that Y_1 is stable by $U \times F$. Then F acts on C so that π is equivariant, and hence π is the desired ruling. \square

Lemma 3.3. *Let T be a torus, U a connected unipotent algebraic group, and Y a variety equipped with a faithful action of $T \times U$. Then there exist a dense open $T \times U$ -stable subset Y_0 of Y and an U -variety Z such that $Y_0 \cong T \times Z$ as $T \times U$ -variety, where $T \times U$ acts on $T \times Z$ via the T -action on itself by multiplication and the U -action on Z .*

Proof. We may replace Y with any $T \times U$ -stable open subset, and hence assume that Y is normal. Next, by results of Sumihiro (see [Su74, Th. 1, Th. 2]), we may assume that Y has an equivariant locally closed embedding in the projectivization $\mathbb{P}(V)$ of a finite-dimensional $T \times U$ -module V . We may further assume that Y is not contained in the projectivization of any proper submodule of V . The dual module V^* contains an eigenvector f of the connected commutative algebraic group $T \times U$. Since Y is not contained in the hyperplane $(f = 0)$, we may replace Y with the complement of this hyperplane, and hence assume that Y is a $T \times U$ -stable open subset of some affine $T \times U$ -variety Y' .

Let $\mathcal{O}(Y') = \bigoplus_{\lambda} \mathcal{O}(Y')_{\lambda}$ be the decomposition of the coordinate ring of Y' into eigenspaces of T , where λ runs over the character group \widehat{T} . Then each $\mathcal{O}(Y')_{\lambda}$ is also U -stable. Since T acts faithfully on the affine variety Y' , the group \widehat{T} is generated by the characters λ such that $\mathcal{O}(Y')_{\lambda} \neq 0$. Thus, we may choose finitely many such characters, say $\lambda_1, \dots, \lambda_N$, which generate \widehat{T} . Then there exist integers a_{ij} , where $1 \leq i \leq N$ and $1 \leq j \leq \dim(T) := r$, such that the characters $\mu_j := \sum_{i=1}^N a_{ij} \lambda_i$ ($j = 1, \dots, r$) form a basis of \widehat{T} . Also, each U -module $\mathcal{O}(Y')_{\lambda_i}$ contains a nonzero U -fixed point, say f_i . Now consider the Laurent monomials

$$g_j := \prod_{i=1}^N f_i^{a_{ij}} \quad (j = 1, \dots, r).$$

These are U -invariant rational functions on Y' , or equivalently on Y . Let Y_0 be the largest open subset of Y on which g_1, \dots, g_r are all regular and invertible. Then the product map

$$h := g_1 \times \dots \times g_r : Y_0 \longrightarrow \mathbb{G}_m^r$$

is a U -invariant morphism; moreover, identifying T with \mathbb{G}_m^r via $\mu_1 \times \dots \times \mu_r$, we see that h is also T -equivariant. Thus, $Y_0 \cong T \times Z$, where $Z := h^{-1}(1, \dots, 1)$, and this isomorphism has the required properties. \square

3.3 Examples

In this subsection, A denotes an abelian variety.

3.3.1 Let G be a semi-abelian variety, extension of A by a torus T . Every such extension is classified by a homomorphism $c : \widehat{T} \rightarrow \text{Pic}^o(A)$, where \widehat{T} denotes the character group of T , and $\text{Pic}^o(A)$ is the dual abelian variety of A . Moreover, G is anti-affine if and only if c is injective (see e.g. [BSU12, Sec. 5.3]).

Next, let Y be a toric T -variety and consider the associated bundle

$$\varphi : X := G \times^T Y \longrightarrow G/T = A.$$

Then X is a normal variety on which G acts with an open orbit having a trivial isotropy subgroup scheme. In particular, $\dim(G) = \dim(X)$.

Assume that X (or equivalently Y) is complete and that G is anti-affine. Then we claim that G is the largest connected automorphism group of X . Indeed, φ is a homogeneous fibration and hence yields a homomorphism

$$\varphi_* : G(X) \longrightarrow A$$

which restricts to the Albanese map $\alpha : G \rightarrow A$. In particular, φ_* is surjective; its kernel K is contained in the group $\text{Aut}_A(X)$ of relative automorphisms. Also, since $G(X)$ contains the anti-affine group G and hence centralizes that group, we see that K is contained in the group of G -equivariant relative automorphisms, $\text{Aut}_A^G(X)$. But $\text{Aut}_A^G(X) \cong \text{Hom}^T(G, \text{Aut}(Y))$. Moreover, since Y is rational, every connected component of $\text{Aut}(Y)$ is an affine variety, and hence every morphism $G \rightarrow \text{Aut}(Y)$ is constant. Thus, $\text{Aut}_A^G(X) \cong \text{Aut}^T(Y)$. But the latter group is just T , since Y contains T as an open orbit. It follows that $K = T$; this yields our claim.

3.3.2 Denote by $E(A)$ the universal vector extension of A . This is a connected commutative algebraic group that sits in an exact sequence

$$0 \longrightarrow V \longrightarrow E(A) \longrightarrow A \longrightarrow 0,$$

where $V := H^1(A, \mathcal{O}_A)^*$ is a vector space of dimension $g := \dim(A)$. Moreover, $E(A)$ is anti-affine if and only if $\text{char}(k) = 0$ (see [BSU12, Sec. 5.4]).

Next, let

$$\pi : \mathbb{F}_{g-1} := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(g-1) \oplus \mathcal{O}_{\mathbb{P}^1}) \longrightarrow \mathbb{P}^1$$

be the rational ruled surface of index $g-1$; then V acts on \mathbb{F}_{g-1} by translations. The associated bundle

$$X := E(A) \times^V \mathbb{F}_{g-1} \longrightarrow E(A)/V = A$$

is again a homogeneous fibration. Moreover, $\dim(G) = 2g$ while $\dim(X) = g+2$, so that $\dim(G) = 2\dim(X) - 4$. Arguing as in the above example, one checks that G is again the largest connected automorphism group of X .

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